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On a recent theorem of M. Shirvani on subgroups of division algebras

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Abstract

In some very nice, recent and yet to be published work Shirvani produces some very tight bounds for the index of an abelian normal subgroup in a soluble-by-finite subgroup of a finite-dimensional division algebra. Here we present an alternative proof of Shirvani's Theorem. With a little more work our method also yields slightly improved bounds. Finally we present examples that restrict further improvements to the bounds and discuss a couple of variants.

Keywords: Soluble group; Finite-dimensional division algebra

Throughout we maintain the following notation: D is a division ring with center F and finite degree $d > 1$, so $\dim_F D = d^2$. Further G always denotes a soluble-by-finite subgroup of the multiplicative group D^* of D.

In his interesting paper [1], amongst much else, Shirvani proves the following.

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1. (Shirvani [1, Theorem 4.1]). The group G contains an abelian normal subgroup A of finite index dividing bcd, where $c < d$ is a divisor of d and b is 1, 6, 12 or 30. Moreover $b = 1$ if either char $D > 0$ or d is odd or G is torsion-free.

An earlier related result of the present author is the following.

2. [3, Theorem 2]. Suppose d is a prime power. Then G has an abelian normal subgroup A of finite index such that with $H = C_G(A)$ we have

either $A = H$ and $(G : A)$ divides d,
or char $D = 0$, $d = 2^m$, $(G : H)$ divides 2^{m-1} and H/A is isomorphic to Alt(4), Sym(4) or Alt(5); hence $(G : A)$ divides bd, where b is 6, 12 or 30,
or char $D = 0$, $d = 2^m \geqslant 4$, $(G : H)$ divides 2^{m-2} and H/A is isomorphic to Sym(5); hence $(G : A)$ divides bd, where $b = 30$.

We present below an alternative proof of Shirvani's Theorem above, based more on the contents of [3,4]. Throughout T denotes the maximal periodic normal subgroup $\tau(G)$ of G. We break the proof into three cases according to the structure of T.

Case I. T has a characteristic metabelian subgroup S with all its Sylow subgroups abelian and with index at most 2 in T.

Case II. Either (a) T is binary octahedral, or (b) $T = Q \times M$, where M is a metabelian $2'$-group with all its Sylow subgroups abelian and Q is quaternion of order 8, or (c) $T \cong \mathrm{SL}(2, 3) \times M$, where M is a metabelian $\{2, 3\}'$-group with all its Sylow subgroups abelian.

Case III. T is isomorphic to $\mathrm{SL}(2, 5)$.

By 2.1.1 and 2.5.9 of [2] all possibilities for T are covered by I, II and III. Moreover if char $D > 0$ then Case I always applies with $S = T$. We prove the following.

3. Theorem. *The group G has an abelian normal subgroup A such that if $e = \dim_F F[A]$, then e divides d and $(G : A)$ divides d^2/e in Case* I*, divides $6d^2/e$ in Case* II *and divides $15d^2/e$ in Case* III.

We also have:

4. Lemma. *In Case* I*, if $e = 1$ then $(G : A) = 1$.*
In Cases II *and* III*, d is even.*

Hence putting 3. and 4. together we obtain in Case I that $(G : A)$ divides $(d/e)d$ and if $e = 1$ then $(G : A) = 1$; in Case II that $(G : A)$ divides $12(d/2)d$; in Case III that $(G : A)$ divides $30(d/2)d$. Thus we have the bounds in Shirvani's Theorem with c being d/e, 1, $d/2$ or $d/2$ and b being 1, 1, 12, or 30, respectively. Further if char $D > 0$ or if d is odd or if a maximal 2-subgroup of G is not quaternion of order 8 or 16 (e.g. if G is torsion-free), then we are in Case I and we have $b = 1$ as claimed in 1.

Our paper below is arranged as follows. First we prove 3. and 4., see Sections 5 and 6 Then in 7. and 8. we discuss the reason for the appearance of d^2 in 3 but only d in its special case 2. In 9. we produce a slight sharpening of the bounds in Shirvani's Theorem 1. (Briefly b can always be taken to be 1, 6 or 15 apart from some very specific configurations with $d = 2$ requiring b to be 12 or 30, all of which are variations on the following groups: binary tetrahedral group ($\cong \mathrm{SL}(2, 3)$), binary octahedral group and binary icosahedral group ($\cong \mathrm{SL}(2, 5)$.) Finally in 10. we give a number of examples where these bounds are attained.

The key result for our proof of 3. is the following special case of part of the Main Theorem from [4].

5. (a) Let S be a characteristic subgroup of T of finite index such that S is metabelian with all its Sylow subgroups abelian. Set $G_0 = C_G(T/S)$. Then G_0 has an abelian characteristic subgroup A such that $F[G_0] \leqslant D$ is a crossed product of $F[A]$ by G_0/A.

(b) If T lies in the second center of G there is an abelian characteristic subgroup A of G such that $F[G] \leqslant D$ is a crossed product of $F[A]$ by G/A.

Part 5.(b) is a special case of part (d) of the Main Theorem of [4] and part 5.(a) is effectively a special case of part (e) of the same result (in this latter case the statement does not contain the precise definition of G_0; this is contained in its proof, specifically it is on lines 9 to 7 from the bottom of [4, p. 301]).

As above set $e = \dim_F F[A]$. It follows at once in 5.(a) that $(G_0 : A)$ divides d^2/e and in 5.(b) that $(G : A)$ divides d^2/e.

6. Proofs of 3. and 4.

Case I. Clearly here $G_0 = C_G(T/S) = G$. By 5.(a) there is an abelian normal subgroup A of G such that $(G : A)$ divides d^2/e. We may choose A to be a maximal abelian normal subgroup of G.

Suppose $e = 1$. Then A lies in the center of G, so G is center-by-finite and therefore $G' \leqslant T$. In Case I the group T has an ascending characteristic series with cyclic factors (see [2, 2.1.1 and 2.5.9] again). Hence if T does not lie in A there exists some x in $T \backslash A$ with $\langle x \rangle A$ abelian and normal in G. This contradicts the maximality of A. Therefore $T \leqslant A$. But then G is nilpotent of class at most 2, so A is a maximal abelian subgroup of G and $A = C_G(A) = G$.

Case II. Set $G_0 = C_G(T/S)$, where S is $\langle -1 \rangle$ in (a) and is $\langle -1 \rangle M$ in (b) and (c). In (a) and (c) let Q denote the normal quaternion subgroup of T of order 8.

Subcase (a). Here $T/\langle -1 \rangle \cong \mathrm{Sym}(4)$ and the outer automorphism group of $\mathrm{Sym}(4)$ is trivial. Thus $TG_0 = G$ and $T \cap G_0 = \langle -1 \rangle$. Also $\tau(QG_0) = Q$, for T, Q and G_0 are all normal in G, so

$$Q \leqslant \tau(QG_0) \leqslant Q.\tau(G_0) = Q(T \cap G_0) = Q.$$

Hence $\tau(QG_0) = Q$ lies in the second center of QG_0. By 5.(b) the algebra $F[QG_0]$ is a crossed product over some abelian characteristic subgroup A of QG_0. Then A is normal in G and $(G:A)$ divides

$$(G:QG_0)d^2/e = (T:Q)d^2/e = 6d^2/e.$$

Subcase (b). Here T/S is a Klein 4-group, so G/G_0 embeds into Sym(3). By 5.(a) there is an abelian characteristic subgroup A of G_0 with $(G_0:A)$ dividing d^2/e. Clearly A is normal in G with $(G:A)$ dividing $6d^2/e$.

Subcase (c). Here T/S is isomorphic to Alt(4), so $T \cap G_0 = S$ and $(G:TG_0) \leqslant 2$, the latter since the automorphism group of Alt(4) is Sym(4). As in the proof of Subcase (a) we have $\tau(QG_0) = Q(T \cap G_0) = QS = QM$. Clearly $S = \langle -1 \rangle \times M$ is characteristic in QM and QG_0 centralizes QS/S. Therefore by 5.(a) the group QG_0 has an abelian characteristic subgroup A with $(QG_0:A)$ dividing d^2/e. Then A is normal in G and $(G:A)$ divides $(G:TG_0)(T:Q)d^2/e$, which divides $6d^2/e$.

Finally we come to Case II of 4. Note that in this case $\dim_F F[Q] \leqslant 4$ and is divisible by a square not 1. Thus $\dim_F F[Q] = 4$ and clearly it divides $\dim_F D = d^2$. Therefore d is even.

Case III. Here $T \cong \mathrm{SL}(2,5)$, so $T/\langle -1 \rangle \cong \mathrm{Alt}(5)$, which is simple. Thus $C_G(T) = C_G(T/\langle -1 \rangle) = G_0$ say. We break the proof for this case into four parts.

(a) If $G = TG_0$, then G_0 has an abelian characteristic subgroup A with $(G:A)$ dividing $15d^2/e$.

Let Q be a Sylow 2-subgroup of T. Then $\tau(QG_0) = Q.\tau(G_0) = Q(T \cap G_0) = Q$, which lies in the second center of QG_0. By 5.(b) there is an abelian characteristic subgroup A of QG_0 with $(QG_0:A)$ dividing d^2/e. Since $(T:Q) = 15$ we have that $(G:A)$ divides $15d^2/e$. What is not yet clear is that A is characteristic in G_0.

Since -1 is the only involution in G and T and G_0 commute with each other and $T \cap G_0 = \langle -1 \rangle$, so $(\mathrm{Aut}\, Q) \times (\mathrm{Aut}\, G_0)$ acts componentwise on $Q \times G_0$ and hence on $QG_0 \leqslant G$. Let $xy \in A$ with $x \in Q$ of order 4 and $y \in G_0$. There is an automorphism ϕ of Q of order 3 that permutes the subgroups of Q of order 4 cyclically. Then $x^{\phi}.y \in A$, so $x^{\phi}.x^{-1} = x^{\phi}.y(xy)^{-1} \in A$. But $x^{\phi}.x^{-1}$ lies in Q and has order 4, so $Q \leqslant A$, which is impossible. Therefore $A \leqslant G_0$. Finally $\mathrm{Aut}\, G_0$ embeds into $\mathrm{Aut}(QG_0)$ as above, so A is also characteristic in G_0 (and in particular is normal in G).

From now on assume that $G \neq TG_0$. Thus now G/G_0 is isomorphic to Sym(5). Choose g in $G \backslash TG_0$ with $g^2 \in G_0$.

(b) $\sqrt{5} \notin F$.

Suppose $\sqrt{5} \in F$ and note that $\mathrm{char}\, F = 0$ in Case III. Now $\mathbf{Q}[T] \leqslant D$ is the quaternion algebra $(-1, -1/\mathbf{Q}(\sqrt{5}))$ over the field $\mathbf{Q}(\sqrt{5})$ where $\mathbf{Q}$ denotes the rationals, see

[2, p. 51, the second paragraph of the proof of 2.1.11]. Then $F[T]$ is the quaternion algebra $(-1,-1/F)$ over $F = F(\sqrt{5})$.

Set $G_1 = T\langle g\rangle$. Now g^2 centralizes T. Then $F[T, g^2]$ is not commutative and has dimension at most 4 over its central subfield $F(g^2)$. Clearly $F[G_1]$ has dimension at most 2 over $F[T, g^2]$, so the dimension of $F[G_1]$ over its center is a square at most 8. It must therefore be 4.

We can now apply 2. above to G_1. Thus G_1 has an abelian normal subgroup A_1 with $(C_{G_1}(A_1) : A_1) \leqslant 60$. But $\langle -1, g^2\rangle$ is the unique maximal abelian normal subgroup of G_1 since $G_1/\langle -1, g^2\rangle \cong \mathrm{Sym}(5)$, so $A_1 \leqslant \langle -1, g^2\rangle$ and

$$\big(C_{G_1}(A_1) : A_1\big) = (G_1 : A_1) \geqslant \big(G_1 : \langle -1, g^2\rangle\big) = 120.$$

This contradiction confirms that $\sqrt{5} \notin F$.

(c) $\dim_F F[T] = 8$, four divides d and the degree of $F[TG_0]$ divides $d/2$.

For $F[T]$ is the quaternion algebra $(-1,-1/F(\sqrt{5}))$ and by (b) we have $\dim_F F(\sqrt{5}) = 2$. Therefore $\dim_F F[T] = 8$. Hence 8 divides d^2, so 4 divides d. Finally $F(\sqrt{5})$ is central in $F[TG_0]$ and $\dim_{F(\sqrt{5})} F[TG_0]$ divides $d^2/2$. Therefore the degree of $F[TG_0]$ divides $d/2$.

(d) G has an abelian normal subgroup A with $(G : A)$ dividing $15d^2/e$.

We may assume that $D = F[G]$. Apply (a) to TG_0, bearing in mind (c). Thus G_0 has an abelian characteristic subgroup A with $(TG_0 : A)$ dividing $15(d/2)^2/e_0$, where e_0 is the dimension over the center E of $F[TG_0]$ of $E[A]$. Also $(G : TG_0) = 2$. Therefore A is normal in G with index dividing $15d^2/2e_0$. Finally $F = C_E(g)$ since $D = F[G]$, so $\dim_F E \leqslant 2$. Hence e is either e_0 or $2e_0$ and the claim follows.

Clearly in Case III, d is even, cf. Case II. The proofs of 3. and 4. are now complete.

7. Remark. The d^2 in 3. is more or less replaced by d in the special case 2. Can we see why? Suppose $d = q^m$ for some prime q. For simplicity assume that $D = F[G]$ and that A is a maximal abelian normal subgroup of G. Then $C_{F[A]}(G) = F$ and $G/C_G(A)$ is isomorphic to the Galois group of $F[A]$ over F. In particular $(G : C_G(A)) = (F[A] : F) = e$.

Consider Case I. By 3. the group G/A is a finite q-group and in particular is nilpotent. By the maximality of A we have $A = C_G(A)$. Thus

$$d^2 \leqslant (G : A)\big(F[A] : F\big) \leqslant e^2.$$

But e divides d. Therefore $e = d$ and $(G : A) = d$. In Cases II and III the integer d is even, so in these cases $q = 2$. To obtain the exact bounds in these cases we would need a more detailed analysis as in the proof of 2. given in [4], making use of the nilpotence of q-groups as in Case I. Incidentally, since we made use of a special case of 2. in our proof of 3., we cannot derive a fully independent proof of 2. from 3.

By weakening slightly the abelian assumption on A in 3. we can easily reduce the d^2 to d. For consider $H = C_G(A)$. The group G/H embeds into the Galois group of $F[A]$ over F and hence $(G : H)$ divides d. Clearly H is central by finite and hence (Schur) is finite by abelian. Further by weakening the normality condition on A in 3. we can also 'replace' the d^2 in 3. by d as the following theorem shows; we make no attempt here to obtain the best possible bounds.

8. Theorem. *The group G has an abelian subnormal subgroup B such that $(G : B)$ divides bd, where b is* 1 *in Case* I, 24 *in Case* II *and* 120 *in Case* III.

Proof. T has a characteristic subgroup S having an ascending characteristic cyclic series with $S = T$ in Case I, with T/S isomorphic to a Klein 4-group or Alt(4) or Sym(4) in Case II and with T/S isomorphic to Alt(5) in Case III. Set $G_0 = C_G(T/S)$. Then $(G : G_0)$ divides b, where b is as in the statement of 8.

Let A be a maximal abelian normal subgroup of G of finite index and set $H = C_G(A)$ and $H_0 = H \cap G_0$. Let $e = \dim_F F[A]$. By Galois theory $(G : H)$ divides e. Consequently $(G : H_0)$ divides be.

H is center-by-finite, so $H' \leqslant T$. Hence $H_0' \leqslant T \cap G_0 = S$, except for (b) of Case II. Suppose H_0' has an ascending cyclic series normal in G, as it does except possibly in Case II(b). Now H_0' centralizes A. By the maximality of A we have $H_0' \leqslant A$. Therefore H_0 is nilpotent of class at most 2. Let B be a maximal abelian subgroup of H_0. Then B is equal to its centralizer in H_0, so $(H_0 : B)$ divides $\dim_{F[A]} F[B] = f$ say. Thus $(G : B)$ divides bef. Further $ef = \dim_F F[B]$, which in turn divides d. Therefore $(G : B)$ divides bd. Finally B is normal in H_0, which is normal in G.

It remains to consider Case II(b) in more detail. Here $T \leqslant G_0$ and T has an ascending cyclic series normal in G_0. Hence the previous case applies to G_0. Consequently G_0 has an abelian 2-step subnormal subgroup B with $(G_0 : B)$ dividing d. (Notice that when we replace G by G_0 the groups T and G_0 remain unaltered.) Thus in these cases too $(G : B)$ divides bd. Clearly B is subnormal in G. □

The following is a slight improvement on Shirvani's Theorem.

9. Theorem. *The group G contains an abelian normal subgroup A of finite index dividing bcd, where $c < d$ divides d (and depends on G and A) and*

- $b = 1$ *in Case* I;
- $b = 6$ *in Case* II, *unless $d = 2$, $D \cong (-1, -1/F)$ the quaternion algebra, $A = G \cap F$, $(G : A) = 24$ and $GF^* = HF^*$ with $H = \langle i, j, i + j, -(1 + i + j + ij)/2\rangle$, where we must take b to be* 12;
- $b = 15$ *in Case* III, *unless $d = 2$, $D = (-1, -1/F)$ the quaternion algebra, $A = G \cap F$, $(G : A) = 60$ and $G = TA$ with $T \cong \mathrm{SL}(2, 5)$, where we must take b to be* 30.

Proof. Again we have to divide into various cases. Case I is already covered by 3. and 4. above.

Case II. Here we have a normal quaternion subgroup Q of G of order 8 and $[Q, S] = \langle 1\rangle$, where S is as in 6. ($S = \langle -1\rangle$ for (a) and $S = \langle -1\rangle M$ for (b) and (c)). Now set $G_0 = C_G(QS/S) = C_G(Q/\langle -1\rangle)$ and $D_0 = F[G_0]$; let F_0 denote the center of D_0 and d_0 the degree of D_0. Clearly $\tau(G_0) = QS$. By 5.(a) there is an abelian characteristic subgroup A_0 of G_0 of index dividing d_0^2/e_0 for e_0 the dimension of $F_0[A_0]$ over F_0. Thus there is an abelian normal subgroup A of G with $(G : A)$ dividing $6d_0^2/e_0$ and maximal with these properties.

If $D \neq D_0$, then $c = d_0$ satisfies $c \mid d$, $c \neq d$ and $(G : A) \mid 6c^2$. Assume $D = D_0$, so now $F = F_0$, $d = d_0$ and $e = e_0$. If $e > 1$ set $c = d/e$. Then $(G : A) \mid 6cd$ as required. Henceforth assume that $e = 1$, so now $A \leqslant F^*$ and A is central in G. Also we may assume that 3 divides $(G : G_0)$ and hence $(G : A)$, for if not $(G : G_0) \leqslant 2$ and T has an ascending cyclic series normal in G. But then $A \geqslant T \geqslant G'$, so G is nilpotent of class at most 2, $A = C_G(A) = G$ and $(G : A) = 1$.

Let $G_1 = C_G(Q)$. Now the centralizer of $Q/\langle -1\rangle$ in $\operatorname{Aut} Q$ is $\operatorname{Inn} Q$. Consequently $G_0 = G_1 Q$. Also $G_1 \cap Q = \langle -1\rangle$ and $\tau(G_1) = S$, which has an ascending characteristic cyclic series. By the maximal choice of A we have $S \leqslant A \leqslant F^*$. Also G is center-by-finite, so $G_1' \leqslant \tau(G_1)$. Therefore G_1 is nilpotent of class at most 2.

Now $D = D_0 = F[QG_1]$ and $Q \leqslant C_D(G_1)$. Therefore $D = C_D(G_1)[G_1]$. Clearly G lies in the normalizer of G_1 in D^*. By [1, Theorem 2.1] there are only three possibilities for this normalizer. Now $\tau(G_1) \cap Q = \langle -1\rangle$, so if $\tau(G_1)$ contained an element x of order 4, then $\langle x\rangle Q$ could not be (generalized) quaternion. Further $\tau(G_1)$ is abelian (even locally cyclic). Thus the first two possibilities for $N_{D^*}(G_1)$ are ruled out and consequently $G \leqslant G_1.C_{D^*}(G_1)$. Therefore $G = G_1.C_G(G_1)$.

By the maximal choice of A, the center of G_1 lies in A. Hence for any y in G_1 we have that $\langle y\rangle A$ is abelian and normal in G. Again by the maximal choice of A we have $G_1 \leqslant A$. But then $(G : A)$ divides $|\operatorname{Aut} Q| = 24$. Always d is even in Case II, see 4. If $d > 2$ we are finished for $(G : A) \mid 6c^2$ with $c = 2$. Let $d = 2$. If $(G : A)$ divides 12 set $c = 1$. If not we have $(G : A) = 24$. (Recall 3 divides $(G : A)$.) Now

$$D = F[G_0] = F[QG_1] = F[Q] \cong (-1, -1/F).$$

Since $(G : A) = 24$, so $(G : G_0) = 6$. If $\sqrt{2} \in F$ there is a copy K of the binary octahedral group in D^* normalizing Q. If $g \in G$ there exists k in K inducing on Q the same automorphism as g. Hence $k^{-1}g \in C_D(Q) = F$. Consequently $GF^* = KF^*$. In general G lies in HF^*, where $H = \langle i, j, i + j, -(1 + i + j + ij)/2\rangle$ and $Q = \langle i, j\rangle$; in fact $GF^* = HF^*$.

Case III. If $D > F[G]$ or if $e > 1$, the claims follow from 3. Thus assume otherwise and assume also that the A given by 3. is chosen to be maximal. Then $A = G \cap F$ and has index in G dividing $15d^2$. The proof splits into two subcases.

(a) Suppose $G = TG_0$ as in Case III, part (a) of 6.

Here $T \cong \mathrm{SL}(2, 5)$, $G_0 = C_G(T)$ and $T \cap G_0 = \langle -1\rangle$. Now G is center-by-finite, so G' is periodic and $G_0' \leqslant T \cap G_0 = \langle -1\rangle$. Thus G_0 is nilpotent of class at most 2. Therefore any maximal abelian subgroup of G_0 is normal in G_0 and hence in $G = TG_0$. Thus $F^* \geqslant A =$

$G_0 \cap C_G(A) = G_0$. Consequently $(G : A) = (T : \langle -1 \rangle) = 60$ and $G \leqslant TF^*$, so $G = TA$. Finally $D = F[G] = F[T] = (-1, -1/F)$ and $d = 2$.

(b) Now suppose $G > TG_0$.

Consider the proof of this case in 6.; specifically consider Case III, part (d) of 6. There we construct an abelian normal subgroup A of G with $(G : A)$ dividing $15d^2/2e_0$ and e_0 is e or $e/2$. But here we have $e = 1$, so $e_0 = 1$ and $(G : A)$ divides $15cd$ for $c = d/2$. Note that d is even, for example, by 4. above. □

10. Examples. We give below a series of examples that show the above results are essentially the best possible. First we give three basic examples, one for each of the three Cases I–III. Then we show how to combine them in various ways to obtain more general examples. The outcome is the following.

Theorem. *In Theorem* 9 *the precise bounds can be attained with the following values of the parameters.*

Case I: (a) *In any characteristic with $c = p$ and $d = 3p$, for p any prime not the characteristic and congruent to* 2 *modulo* 3. (*Note that there are infinitely many primes p with these properties.*)

(b) *In any characteristic not* 2 *with $c = 2^r$ and $d = 2^r 3^r r$, where r is any positive integer.*

Cases II *and* III: (a) *The special cases for $d = 2$.*

(b) *With $c = 2^{r+1}$ and $d = 2^{r+1}3^r r$, where r is any positive integer.*

Example (i) for Case I. Let $N = \langle x, y, z \mid z = [x, y], [x, z] = [y, z] = z^p = 1\rangle$. Here p is any prime congruent to 2 modulo 3. Let $a = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z})$; so $|a| = 3$. Let b have infinite order and act on N via a; specifically let $x^b = x^{-1}y^{-1}$ and $y^b = x$. Then $z^b = z$. Set $G = \langle b\rangle N$, the split extension of N by $\langle b\rangle$. The center Z of N is $\langle x^p, y^p, z\rangle$ and N/Z is an elementary abelian p-group of rank 2. Also $G/\langle z\rangle$ is poly infinite-cyclic, b acts irreducibly on N/Z (since $\mathrm{GF}(p)$ has no primitive cube root of 1 by the choice of p) and $\langle b^3, z\rangle$ is central in G.

Let A be a maximal abelian normal subgroup of G. Then $\langle b^3, z\rangle \leqslant A$. Also $(A \cap N)Z$ is abelian while N is not, so $(A \cap N)Z \neq N$. Thus since N/Z is $\langle b\rangle$-irreducible we have $A \cap N \leqslant Z$. Now clearly $\langle b^3\rangle Z$ is an abelian normal subgroup of G and $\langle b^3\rangle N/\langle b^3\rangle/Z$ is the unique maximal abelian normal subgroup of $G/\langle b^3\rangle Z$. Thus $A \leqslant \langle b^3\rangle N$, so $A = \langle b^3\rangle(A \cap N) \leqslant \langle b^3\rangle Z$. Hence $A = \langle b^3\rangle Z$ and in particular $(G : A) = 3p^2$. Also the center of G is $C_A(b) = \langle b^3\rangle C_Z(b)$.

Let P be a prime field of characteristic not p, let ζ be a primitive pth root of 1 over P and set $E = P(\zeta)$. Let R be the crossed product of E by $G/\langle z\rangle$ after identifying ζ and z. Then R is an Ore domain [2, 1.4.3]; let D denote its division ring of quotients and F the center of D. Trivially $G \leqslant D^*$.

Now D has dimension $3p^2$ over $E(b^3, Z)$ and b does not centralize Z, so $E(b^3, Z)$ has dimension 3 over the centralizer C of b in $E(b^3, Z)$. Clearly $C \leqslant F$ and $\dim_C D = 3^2 p^2$. Now A is not central in G, so $e = \dim_F F[A] = 3$. By 3 and the uniqueness of A we have

$$3p^2 = (G : A) \leqslant d^2/3 \leqslant 3^2 p^2/3$$

for $d = \deg D$. Therefore $d = 3p$ (and $F = C$).

Thus we have constructed an example in Case I for any characteristic with $d = 3p$, $e = 3$ and $(G : A) = d^2/e$; here p is any prime other than the characteristic congruent to 2 modulo 3. (Of course if $e = 1$ then $(G : A) = 1$ by 4. and consequently $(G : A) = d^2$ is obtainable only in the trivial case $d = 1$.)

Example (ii) for Case II. Let $D = (-1, -1/\mathbf{Q})$ and set $H = \langle i, j, i + j, -(1+i+j+ij)/2\rangle \leqslant D^*$. Then $\langle i, j, -(1 + i + j + ij)/2\rangle \cong \mathrm{SL}(2, 3)$, $A = \langle -1, 2\rangle$ is central in H and is the unique maximal abelian normal subgroup of H and $H/A \cong \mathrm{Sym}(4)$. Thus here $d = 2$, $e = 1$ and $(H : A) = 24 = 6d^2$. If $D = (-1, -1/\mathbf{Q}(\sqrt{2}))$, then $K = \langle i, j, (i + j)/\sqrt{2}, -(1 + i + j + ij)/2\rangle$ is binary octahedral and $A = \langle -1\rangle$ is the unique maximal abelian normal subgroup of K. We have $K/A \cong \mathrm{Sym}(4)$ and again $d = 2$, $e = 1$ and $(K : A) = 24 = 6d^2$.

H here is covered under Case II(c) and K under Case II(a). The largest value for b we can achieve in Case II(b) for $d = 2$ is 6 and this value is possible (in the exceptional situation of Case II, we have $G/A \cong \mathrm{Sym}(4)$, so $G' \leqslant T$ involves Alt(4) and hence Case II(b) cannot arise). For let $J = \langle i, j, -(1 + i + j + ij)\rangle$. Then $\tau(J) = \langle i, j\rangle = Q$, so J is in Case II(b). Also $A = \langle -1, 8\rangle$ is the unique maximal abelian normal subgroup of J with $J/A \cong \mathrm{Alt}(4)$ and we have $d = 2$ (so c must be 1) $e = 1$ and $(J : A) = 12 = 6cd$.

Example (iii) for Case III. $D = (-1, -1/\mathbf{Q}\sqrt{5})$ contains a copy L of SL(2, 5), $A = \langle -1\rangle$ is the unique maximal abelian normal subgroup of L and $L/A \cong \mathrm{Alt}(5)$. Here $d = 2$, $e = 1$ and $(L : A) = 60 = 15d^2$. Clearly L falls under Case III.

Example (iv) for Case II. Let D and G be as in Example (i) with $p = 2$ and $P = \mathbf{Q}$. Here $\zeta = -1$, so $E = \mathbf{Q}$. Set $D_0 = (-1, -1/E)$. Using [2, 1.4.3] form the division ring D_1 of quotients of the crossed product of D_0 by $G/\langle z\rangle$ with $z = -1$. Then D_1^* contains a copy of GH, where H is as in Example (ii), with $[G, H] = \langle 1\rangle$ and $G \cap H = \langle -1\rangle$. The subgroup GH has a unique maximal abelian normal subgroup A and $(GH : A) = 2^2.3.24$. If d_1 denotes the degree of D_1 and if e_1 denotes the dimension of the subalgebra of D_1 generated by A over the center of D_1, which is in fact 3, then by 3. we have $2^2.3.24 \leqslant 6d_1^2/3$. But $d_1^2 \leqslant 2^2.(2.3)^2$. Hence $d_1 = 2^2.3$ and $(GH : A) = 2^2.3.24 = 6d_1^2/e_1$. Of course GH falls into Case II.

Example (v) for Case III. Again consider Example (i) with $p = 2$ and $P = \mathbf{Q}$, but now set $E = \mathbf{Q}(\sqrt{5})$. Repeat the construction of Example (i). Everything goes through and we obtain G and A as before with $(G : A) = 2^2.3$. Putting this to one side for the moment now repeat the construction of D_1 in Example (iv), but with this new choice of E. Then $D_0 = (-1, -1/E)$ contains a copy of $L \cong \mathrm{SL}(2, 5)$ and we obtain a subgroup GL of D_1^*

with a unique maximal abelian normal subgroup A satisfying $(GL : A) = 2^2.3.60$. Again we have $d_1 = 2^2.3$, $e_1 = 3$ and $(GL : A) = 15d_1^2/e_1$; clearly GL comes under Case III.

Example (vi) for Case I. Let r be any positive integer and let G be as in Example (i), again with $p = 2$. Let G_r be the central product of r copies of G, amalgamating the copies of z, and extended by an infinite cycle permuting the r copies of G cyclically. Then G_r has a unique maximal abelian normal subgroup A_r with G_r/A_r isomorphic to the wreath product $(G/A)\,\mathrm{wr}(\mathbf{Z}/r\mathbf{Z})$. Then $(G_r : A_r) = 2^{2r}.3^r r$.

Let P be any prime field of characteristic not 2. Identify $z \in G_r$ with -1 in P. Form the crossed product of P by $G_r/\langle z\rangle$ and obtain (via [2, 1.4.3]) a division ring D, with center F say, generated by G_r. Then $e = \dim_F F[A_r] = 3^r r$ and $d^2 \leqslant 2^{2r}.3^r r.3^r r$. By 3. we have $(G_r : A_r) \leqslant d^2/e$. Hence we have equality throughout and $(G_r : A_r) = d^2/e$ for $d = 2^r.3^r r$ and $e = 3^r r$. This group is covered by Case I.

Example (vii) for Case II. Take G_r and D as in Example (vi), but with $P = \mathbf{Q}$. Repeat the construction of Example (iv) with G_r in place of G and obtain a division ring D_1 and a subgroup $G_r H$ of D_1^* covered by Case II such that $G_r H$ has a unique maximal abelian normal subgroup A with $(G_r H : A) = 2^{2r}.3^r r.24$. Also $d_1 = 2.2^r.3^r r$, $e_1 = 3^r r$ and $(G_r H : A) = 6d_1^2/e_1$.

Example (viii) for Case III. If we combine Examples (vi) and (iii) as in Example (v), cf. Example (vii), we obtain a division ring D_1 and a subgroup $G_r L$ of D_1^*, for $L \cong \mathrm{SL}(2,5)$, with a unique maximal abelian normal subgroup A of index $(G_r L : A) = 2^{2r}.3^r r.60$. Further $d_1 = 2.2^r.3^r r$, $e_1 = 3^r r$ and $(G_r L : A) = 15d_1^2/e_1$. Of course $G_r L$ comes under Case III.

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